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Examples of metric spaces

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1. If R be the set of all real numbers then the mapping $d: R\times R \to R$ defined as

$d(x,y) = |x-y| \ \forall\ x,y \in R$ is a metric on R.

Proof: Let us ~~verify~~ verify whether d satisfies the conditions of a metric space.

1. $d(x,y) = |x-y|$

since $|x-y| \geq 0 \ \forall\ x,y \in R \Rightarrow d(x,y) \geq 0 \ \forall\ x,y \in R$.

2. $d(x,y) = 0 \Leftrightarrow |x-y| = 0 \Rightarrow x-y=0 \Rightarrow x=y$

$\therefore d(x,y) = 0 \Leftrightarrow x = y \ \forall\ x,y \in R$

3. $d(x,y) = |x-y|$

but $|x-y| = |y-x|$

$\therefore d(x,y) = |y-x| = d(y,x)$

4. $d(x,y) = |x-y| = |(x-z)+(z-y)|$

$\leq |x-z| + |z-y|$

$\therefore d(x,y) \leq d(x,z) + d(z,y)$

So d satisfies all the conditions $\therefore$ d is a metric on R.

- d is also called USUAL METRIC ON R.

2\. Let X be a non-empty set and d is defined as $d: X \times X \to R$ by

$$d(x,y) = 0, \quad x=y$$
$$= 1, \quad x \neq y.$$

Then d is a metric on X, also called $\boxed{\text{DISCRETE / TRIVIAL METRIC}}$

Let us verify the conditions.

1\. Given that $d(x,y) \geq 0 \ \forall x,y \in X$

2\. Given that
$d(x,y)=0$ when $x=y$

$\Rightarrow d(x,y)=0 \Leftrightarrow x=y$

3\. ~~$d(x,y)$~~ we've to prove that $d(x,y)=d(y,x)$.

If $x=y$, $d(x,y)=0$. If $y=x$, $d(y,x)=0$.
$\therefore d(x,y)=d(y,x)$ when $x=y$ —— (i)

If $x \neq y$, $d(x,y)=1$ (given)
Similarly $y \neq x$, $d(y,x)=1$.

$\therefore d(x,y)=d(y,x)$ when $x \neq y$. —— (ii)

From (i) and (ii) $d(x,y)=d(y,x)$

3rd condition satisfied.

4\. We have to prove that
$d(x,y) \leq d(x,z)+d(z,y) \quad \forall x,y,z \in X$

Also $d(x,y) = 0$ when $x = y$

and $d(x,y) = 1$ when $x \neq y$.

We can choose x, y, z in the following manners, i.e.

(a) $x = y, y \neq z$ (b) $x = y, y = z$

(c) $x \neq y, y = z$ (d) $x \neq y, y \neq z$

(a) $x = y, y \neq z \Rightarrow d(x,y) = 0, d(y,z) = 1, d(x,z) = 1$

$\therefore d(x,y) \leq d(x,z) + d(z,y)$

i.e. $0 \leq 1 + 1$ which is true.

(b) $x = y, y = z \Rightarrow d(x,y) = 0, d(y,z) = 0, d(z,y) = 0$

$\therefore d(x,y) \leq d(x,z) + d(z,y)$

i.e. $0 \leq 0 + 0$ which is true

(c) $x \neq y, y = z \Rightarrow d(x,y) = 1, d(y,z) = 0,$ $d(x,z) = 1$

$d(x,y) \leq d(x,z) + d(z,y)$

i.e. $1 \leq 1 + 0$ which is true.

(d) $x \neq y, y \neq z \Rightarrow d(x,y) = 1, d(y,z) = 1, d(x,z) = 1$

$d(x,y) \leq d(x,z) + d(z,y)$

i.e. $1 \leq 1 + 1$ which is true.

Hence the 4th condition is also satisfied.

So, d is a metric on X.